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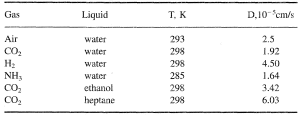
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Diffusion Equation Analysis

# Analytical Solution of the 1-D diffusion Equation

Starting with the known relationship:

Where C = C(x,t) is a function which produces the concentration of a substance at a given time t, and at a given point x on a one-dimensional line, and D is the constant of diffusion for a given substance through a given medium at a given temperature. Some diffusion constants for different substances and mediums are listed below:



To find a solution for C we will use the method of separation of parameters. However, this requires us to assume that a solution will have the form:

C(x,t) = X(x)T(t) so C = XT

Which will likewise give us a partial differential equation of the form:

=> =>

Since X is a function of only x and T is a function of only t we can rewrite this equation so that each side is strictly a function of one variable:

The above equation is fine except for one thing, how can the left side which is only dependent on t equal the right side which is only dependent on X? The answer is that this is only possible if both sides are equal to some constant for all values of x and all values of t, we’ll call this constant -λ2, though this seems somewhat arbitrary it’ll make our solution nicer. This gives us the equation of:

Using the equation above we can form the following system of linear differential equations.

Let’s solve first. Our characteristic equation has the form

There are three cases we must consider:

It turns out that is the only case the lies within the realm of physicial possibility so we may assume that a solution to X will look like:

Under the assumption that solving for T becomes trivial suffice it to say then that:

Now we have a general solution to the diffusion equation, all that remains is to apply our boundary conditions: at x = 0, and at X = L, where L is a measure of the depth of the skin.

Since , we can assume that X’(0) = 0

Since B = 0 the revised version of X will be:

Likewise Since C(L,t) = X(L)T(t) = 0, we can assume that X(L) = 0

For the above equation to be true then for each n.

Our final solution for C will be:

Likewise any summation of will also be a solution:

# Code For plotting the 1-D system



# Plot of 1-D diffusion equation



# Analytical Solution of the 2-D diffusion equation

Now we will use similar techniques as we did with the 1-D diffusion to find a solution for the 2-D diffusion equation.

Starting with the known relationship:

Where u is a measure of concentration of a substance at a given point in time and space, t is a point in time, and x,y are points in space.

As we did before we will assume a solution can be separated into functions of individual variables, first we will separate U into functions of time and space:

Where V is strictly a function of two dimensional space and T is strictly a function of time.

Recalling the original differential equation

=> => =>

Again we have two equations that are dependent on different variables yet equal to each other so both equations must be equal to some constant λ

We can go ahead and solve for T since it is identical to the 1-D case

Now we will assume that V can be split into to two equations of x and y respectively

So now we have

Again for this relationship to be true both and must be equal to some constants for all values of x and y.

So we have the following two differential equations:

We may assume that since in the 1-D case the general solution to the X equation was of the form

That the general solution for both X and Y in the 2-D case will have a similar form.

Now we must find a solution that fits our boundary conditions which are:

at x = 0 so

at x = L so

at y = 0 so

at y = W so

First we will solve for X

Now we will solve for Y

Our final solution for the diffusion equation with the given boundary conditions is: